IRREDUCIBLE HOLOMORPHIC SYMPLECTIC MANIFOLDS

LISA MARQUAND

Abstract. These notes are lecture notes for a series of two lectures given following the paper *Variétés Kāhleriennes dont la Premiére classe de Chern est nulle* by Arnaud Beauville. The goal of the lectures was to show that $S^{[n]}$ are examples of an irreducible holomorphic symplectic manifold, where S is a $K3$ surface. For more details and references, we refer the reader to Beauville's original paper.

1. Introduction

Our main goal is to understand Beauville's construction of examples of irreducible holomorphic symplectic manifolds. We will first try to motivate the existence of such manifolds, illustrating them as one of the three building blocks that make up compact, complex Kāhler manifolds with trivial first chern class. We will review examples of holomorphic symplectic manifolds,especially in the compact dimension 2 case. Our next goal will be to exploit the geometry of the Hilbert scheme of n points to construct a holomorphic symplectic structure on $S^{[n]}$ when S is a surface with trivial canonical bundle. In particular, when S is a $K3$ surface, we will show that such a structure is unique, and that $S^{[n]}$ is simply connected.

1.1. **Motivation and definition.** Let X be a compact, Kāhler manifold with $c_1(X) = 0$. A famous result is that X is made up of three building blocks (up to a finite cover). We have the following structure theorem:

Theorem 1.1 (Structure Theorem)**.** *Let X be as above. Then:*

(1) *The universal cover $\widetilde{X}$ of X is*
$$\widetilde{X} \cong \mathbb{C}^k \times \prod V_i \times \prod X_j,$$
where
 (a) *V_i is a projective, simply connected manifold of dimension at least 3, with $K_{V_i} \cong \mathcal{O}_{V_i}$ and $H^0(V_i, \Omega^p) = 0$ for $0 < p < m_i$.*
 (b) *X_j is an irreducible holomorpic symplectic manifold.*
This decomposition is unique up to ordering.

(2) *There exists a finite étale covering $X' \to X$ where*
$$X' \cong T \times \prod V_i \times \prod X_j$$
where T is a complex torus.

The V_i in the decomposition are sometimes referred to as strict Calabi-Yau; we won't mention them in the rest of these notes.

Date: April 7, 2022.

Let us now define irreducible holomorphic symplectic manifolds - this will follow after several definitions. Let X be a complex manifold of dimension $2n$, and T_X denote the holomorphic tangent bundle of X. Recall that a holomorphic 2-form ω on X is **nondegenerate** at a point $p \in X$ if the alternating form $\omega(p)$ is non-degenerate on T_pX. We see that $\omega^n(p)$ is no-zero at p.

Definition 1.2. Let X be a complex manifold. A holomorphic symplectic structure on X is a closed, holomorphic 2-form on X that is non-degenerate at every point. We usually denote this form by ω. We say that X is a holomorphic symplectic manifold.

Remark 1.3. The existence of such a structure implies that the dimension of X is even, i.e $\dim_{\mathbb{C}} X = 2n$, and the canonical bundle is trivial, $K_X \cong \mathcal{O}_X$.

Remark 1.4. Let $B \subset X$ be a submanifold of codimension at least 2. Then any holomorphic symplectic structure ω on $X \setminus B$ extends uniquely to a holomorphic symplectic structure on X. Indeed, by Hartog's theorem ω extends to a 2-form $\bar{\omega}$ on X. The divisor of $\bar{\omega}^n$ is contained in B, thus must be 0, and hence $\bar{\omega}$ is non-degenerate.

Let us exhibit some examples of holomorphic symplectic manifolds:

(1) The cotangent bundle of any complex ariety T^*X is holomorphic symplectic.
(2) Let Y be a complex manifold of dimension $2n$, and ω any closed 2-form on Y such that $\omega^n \neq 0$. Then ω induces a holomorphic symplectic structure on $Y \setminus div(\omega^n)$.
(3) The only compact, 2-dimension examples are complex tori, and $K3$ surfaces.

Definition 1.5. Let X be a compact, kāhler manifold. We say that X is a **irreducible holomorphic symplectic manifold** if:

(1) there exists a unique holomorphic symplectic structure ω on X.
(2) X is simply connected.

Remark 1.6. The definition is motivated by differential geometry, and has an equivalent definition: X admits a kāhler metric whose holonomy group is $Sp(n)$.

A $K3$ surface is thus the only example in dimension 2. Our main goal is to construct examples of irreducible holomorphic symplectic manifolds in higher dimensions.

2. The Hilbert Scheme of n points

In this section, we let S be any compact projective surface.

2.1. **The symmetric product.** Let $S^n := S \times \cdots \times S$ be the product of n copies of a surface S. Consider the natural quotient map $\pi : S^n \to S^{(n)} := S^n/\Sigma_n$, where Σ_n is the symmetric group of n elements, acting on S^n by permuting the factors. The variety $S^{(n)}$ is singular, with quotient singularities. We will be interested in the local structure.

Example 2.1. Let us compute $(\mathbb{C}^2)^2$. The symmetric group is just generated by an involution ι switching the two factors: $\iota : (a, b), (c, d) \mapsto (c, d)(a, b)$. Consider the change of co-ordinates:

$$\begin{aligned} w &= a + c \\ x &= b + d \\ y &= a - c \\ z &= b - d. \end{aligned}$$

In these new co-ordinates, the involution is given by $\iota : (w, x, y, z) \mapsto (w, x, -y, -z)$, and so $\iota = id_{\mathbb{C}^2} \times (-id_{\mathbb{C}^2})$. We write the ring of regular functions on $(\mathbb{C}^2)^(2)$ as the invariant part of the ring of regular functions on $\mathbb{C}^2 \times \mathbb{C}^2$, we see that this is:

$$\mathbb{C}[w, x, y^2, z^2, yz] \cong \mathbb{C}[w, x] \otimes \mathbb{C}[u, v, t]/(t^2 - uv).$$

Thus we can identify $(\mathbb{C}^2)^{(2)}$ with $\mathbb{C}^2 \times Q$, where Q is the quadric cone given as $\{t^2 - uv = 0\} \subset \mathbb{C}^3$.

2.2. **The Hilbert Scheme of n points of a surface.** We will now give a somewhat vague definition of the Hilbert scheme $X^{[n]}$; for our purposes, we will only require certain facts about this scheme which we will soon state.

Definition 2.2. Let X be a smooth projective variety. Then $X^{[n]}$ is a variety that parametrises the subschemes of X of length n.

Let S now be a smooth projective surface. We will use several facts - we will try to explain and motivate as we go.

(1) $S^{[n]}$ is an irreducible, smooth variety of dimension $2n$, called the Hilbert scheme of n points on S.
(2) There exists a morphism $\epsilon : S^{[n]} \to S^{(n)}$ called the Hilbert-Chow morphism, maps a subscheme
$$Z \mapsto \sum_{p \in S} length_p(Z)p.$$
Moreover, the map ϵ is a resolution of singularities of $S^{(n)}$.
(3) Let
$$\Delta_{ij} := \{(x_1, \ldots x_n) \in S^n \mid x_i = x_j\} \subset S^n,$$
and let $\Delta := \cup_{i,j} \Delta_{ij}$. Then $S^{(n)}$ is singular along $D := \pi(\Delta)$, and $\epsilon^{-1}(D)$ is an irreducible divisor E.

Example 2.3. Let us consider $S^{[2]}$. Notice that $\mathrm{Sing}(S^{(2)}) = \{2x \mid x \in S\} = D$. In this case, one can define directly: $S^{[2]} = Bl_D S^{(2)}$.

(4) Consider the subset $D_* \subset D$, where
$$D_* := \{2x_1 + x_2 + \cdots + x_{n-1} \mid x_i \text{ are distinct }\}.$$
Let
$$S_*^{(n)} := (S^{(n)} \setminus D) \cup D_*,$$
where we have removed the singular locus, and added back in the locus parametrising 2-collisions. Put
$$S_*^{[n]} := \epsilon^{-1}(S_*^{(n)}), \;\; S_*^n : +\pi^{-1}(S_*^{(n)}).$$
Condition (3) ensures that $S^{[n]} \setminus S_*^{[n]}$ is a closed subspace of $S^{[n]}$ of codimension 2.

By Remark 1.4, in order to construct a holomorphic symplectic structure on $S^{[n]}$, it suffices to construct such a structure on $S^{[n]}_*$, which has been designed to be easier to work with. In order to see this, we shall discuss the restricted Hilbert-Chow morphism

$$\epsilon : S^{[n]}_* \to S^{(n)}_*.$$

(5) Notice that $S^{(n)}_*$ has two strata - the smooth locus parametrising n distinct points $X_1 + \cdots + x_n$, and the open locus where two of these points collide: $(2x_1 + \ldots x_{n-1})$. Locally near one of the points $2x_1 + x_2 + \ldots x_{n-1}$, we can identify $S^{(n)}_*$ with

$$(\mathbb{C}^2)^{(2)} \times \mathbb{C}^{(2n-2)}.$$

From Example 2.1, we can conclude that $(S^{(n)}_*, D_*)$ is locally isomorphic to $(Q \times \mathbb{C}^{2n-2}, q \times \mathbb{C}^{2n-2})$, where $q \in Q$ is the vertex of the cone.

(6) The map $\epsilon : S^{[n]}_* \to S^{(n)}_*$ is identified with the blow up of D_* in $S^{(n)}_*$.

(7) Let $\Delta_* := \Delta \cap S^n_*$; notice that this is now smooth of codimension 2. Let

$$\eta : Bl_{\Delta_*} S^n_* \to S^n_*$$

be the blow up of Δ_*; denote by $E_{ij} = \eta^{-1}((\Delta_{ij})_*)$ the exceptional divisors. Notice that the action of Σ_n extends to $Bl_{\Delta_*} S^n_*$. We can identify

$$Bl_{\Delta_*} S^n_* / \Sigma_n = S^{[n]}_*,$$

and we have the following diagram:

$$\begin{array}{ccc} Bl_{\Delta_*} S^n_* & \xrightarrow{\eta} & S^n_* \\ \downarrow \rho & & \downarrow \pi \\ S^{[n]}_* & \xrightarrow{\epsilon} & S^{(n)}_* \end{array} \tag{2.4}$$

3. $S^{[n]}$ is an irreducible holomorphic symplectic manifold

Our main goal of this section is to prove that $S^[n]$ is an irreducible holomorphic symplectic manifold when S is a $K3$ surface. First, we will show that if S has trivial canonical bundle, then there exists a holorphic symplectic structure on $S^{[n]}$. Following this, we will relate $H^2(S^{[n]}, \mathbb{C})$ to that of $H^2(S, \mathbb{C})$; using this, we shall show this structure is unique when S is a $K3$ surface. We will finally state a topological result which allows us to conclude that $S^{[n]}$ is in this case simply connected.

3.1. **The holomorphic symplectic structure.**

Proposition 3.1. *Let S be a complex projective surface with $K_S \cong \mathcal{O}_S$. Then $S^{[n]}$ admits a holomorphic symplectic structure.*

Proof. We will construct a nondegenerate holomorphic 2-form on $S^{[n]}_*$. By Remark 1.4, it will extend to a holomorphic symplectic structure on $S^{[n]}$.

Let τ be a nondegenerate holomorphic 2-form on S, the existence of which is guaranteed by the trivial canonical bundle condition. Let $\tilde{\tau} = \sum_{i=1}^n pr_i^* \tau$, where $pr_i : S^n \to S$ is the projection to the ith factor. This defines a nondegenerate holomorphic 2 form on S^n_*. Consider $\eta^* \tilde{\tau}$ on $Bl_{\Delta_*} S^n_*$. Since $\tilde{\tau}$ is invariant under the action of Σ_n, the same holds for $\eta^* \tilde{\tau}$. Since $S^{[n]}_*$ is the quotient of $Bl_{\Delta_*} S^n_*$ by this finite group action, the invariant 2-forms of $Bl_{\Delta_*} S^n_*$ are identified with the

2-forms of $S_*^{[n]}$. In particular, there exists a holomorphic 2-form ω on $S_*^{[n]}$ such that

$$\eta^*\tilde{\tau} = \rho^*\omega.$$

It remains to show that ω is non-degenerate. The map ρ is a ramifies covering that is ramified along the exceptional divisors E_{ij}. Thus we have that

$$div(\rho^*\omega^n) = \rho^* div(\omega^n) + \sum E_{ij}.$$

On the other hand, we know that

$$div(\rho^*\omega^n) = div(\eta^*\tilde{\tau}) = \sum E_{ij}.$$

It follows that $div(\tau^n) = 0$, and extending τ defines the holomorphic symplectic structure on $S^{[n]}$. □

3.2. **Uniqueness of the holomorphic symplectic structure.** Recall that for a smooth projective variety X,

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

where $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$.

Lemma 3.2. *Let S be a compact projective surface, $n \geq 2$.*

(1) *The induced map*

$$\epsilon^* : H^2(S^{(n)}, \mathbb{C}) \to H^2(S^[n], \mathbb{C})$$

is injective on Hodge structures, and we have that

$$H^2(S^{[n]}, \mathbb{C}) = \mathrm{Im}\epsilon^* \oplus \mathbb{C}[E].$$

(2) *The map $\pi^* : H^2(S^{(n)}, \mathbb{C})) \to H^2(S^n, \mathbb{C})$ induces an isomorphism*

$$H^2(S^{(n)}, \mathbb{C}) \cong H^2(S^n, \mathbb{C})^{\Sigma_n}.$$

(3) *If further $H^1(S, \mathbb{C}) = 0$, then π^* induces an isomorphism of Hodge structures of*

$$H^2(S^{(n)}, \mathbb{C}) \cong H^2(S, \mathbb{C}).$$

Proof. (2) Since $S^{(n)}$ is the quotient of a projective variety by a finite group, it satisfies Poincaré duality, and has a pure Hodge structure. This leads to the assertion.

(1) We again replace $S^{[n]}$ by $S_*^{[n]}$, and S^n by S_*^n; this does not modify the second cohomology. Consider the diagram induced by 3.3:

$$\begin{array}{ccc} H^2(Bl_{\Delta_*} S_*^n)^{\Sigma_n} & \xleftarrow[\eta^*]{} & H^2(S_*^n)^{\Sigma_n} \\ \rho^* \uparrow & & \pi^* \uparrow \\ H^2(S_*^{[n]}) & \xleftarrow[\epsilon^*]{} & H^2(S_*^{(n)}) \end{array} \tag{3.3}$$

where ρ^*, π^* are bijective. Since η is a blow up of a smooth variety in a smooth center, we see that

$$\begin{aligned} H^2(Bl_{\Delta_*} S_*^n)^{\Sigma_n} &\cong \mathrm{Im}\eta^* \oplus \left(\sum \mathbb{C}[E_{ij}]\right)^{\Sigma_n} \\ &\cong \mathrm{Im}\eta^* \oplus \mathbb{C}[E]. \end{aligned}$$

(3) One can use the Kūnneth formula to expand $H^2(S^n, \mathbb{C})$ in terms of $H^2(S, \mathbb{C})$ due to the vanishing of $H^1(S, \mathbb{C})$. From this one can compute the invariant piece under the action of Σ_n to see the result.

□

Corollary 3.4. *Let S be a $K3$ surface. Then there is a unique holomorphic symplectic structure (up to scaling) of $S^{[n]}$.*

Proof. By Lemma 3.2, we see that

$$H^{2,0}(S^{[n]}) \cong H^{2,0}(S) \cong \mathbb{C}[\tau],$$

where τ is the unique holomorphic symplectic form on S. □

3.3. **Simply connectedness.** One can use the topological lemma below to conclude that $S^{[n]}$ is simply connected, we state for completeness.

Lemma 3.5. *Let S be any projective surface, $n \geq 2$.*

(1) *The induced map $\epsilon_* : \pi_1(S^{[n]}) \to \pi_1(S^{(n)})$ is bijective.*
(2) *The induced map $\pi_* : \pi_1(S^n) \to \pi_1(S^(n))$ is surjective, and $\pi_1(S^{(n)})$ is isomorphic to the largest abelian quotient of $\pi_1(S)$.*